Clarke taxes

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September 18, 2011

1 The setting

There are n individuals, $i = 1 \dots n$.

The amount g of a public good, with constant marginal cost c, to be provided has to be determined. Every individual has her private utility $U_i(g)$ in the amount of public good provided.

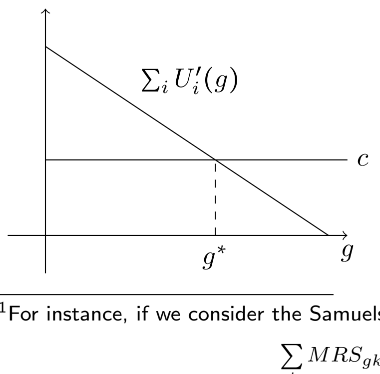
Let's assume that each U_i is well shaped: in particular, that

- it is (increasing and) concave: $U_i' < 0$,
- $U_i'(0) = +\infty$
- $U_i'(+\infty) = 0$.

These conditions then apply also to the sum of the utilities, $\sum_i U_i$. So there is a point g^* such that

$$\sum_i U_i'(g^*) = c; \tag{1}$$

the Samuelson condition tells us that this g^* is the optimal allocation, the one maximizing social welfare.[1]



[1]For instance, if we consider the Samuelson condition in the form

$$\sum_i MRS_{gk}(g^*) = MRT_{gk}(g^*)$$

and we consider as private good k just wealth.

2 The Wicksell problem

Suppose you are the government. You know that you should produce a g^* which is optimal given the preferences of the citizens.

You ask them "tell me your preferences", each one of them reveals $\hat{U}_i$, then a point $\hat{g}^*$ such that

$$\sum_i \hat{U}_i'(\hat{g}^*) = c$$

is chosen. Then, you subdivide the payment by consequence: each individual i pays $\hat{U}_i(g^*)$.

All nice, if indeed the citizens declared $\hat{U}_i = U_i$. Unfortunately, a single individual has an incentive to lie, and declare $\hat{U}_i(g) < U_i(g)\ \forall g$. For instance, she can declare $\hat{U}_i(g) \equiv 0$: she will be required to pay nothing at all, and (assuming for instance that the others report U_i truthfully) she will still benefit of some (lower, but positive) quantity of public good: there is a freeriding problem.
So the optimal solution cannot be naively implemented in this way: we need an incentive compatible mechanism to make truthful revelation convenient.

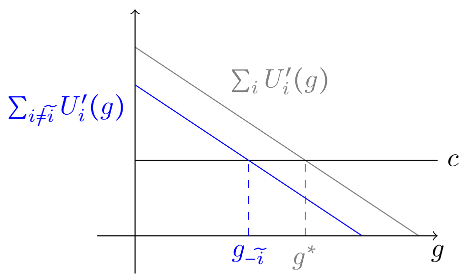
3 The Clarke mechanism - simplest version

The Clarke mechanism does precisely that: it envisages a system of individual transfers from the citizens to the government such that each citizen's optimal strategy is to report $\hat{U}_i = U_i$. Then, once government will know that the declarations of citizens are true, it will be able to establish what is the optimal quantity of public good g^*.

Let's consider a particular individual $\tilde{i}$, which declares her utility from any quantity of public good g being $\hat{U}_g$.

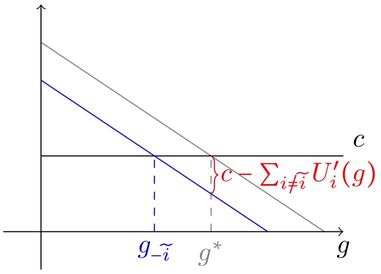
Given the properties of the utility functions, we can find a value of public good $g_{-\tilde{i}}$ such that

$$\sum_{\forall i \neq \tilde{i}} \hat{U}_i'(g_{-\tilde{i}}) = c.$$



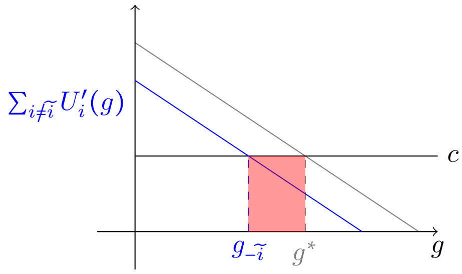
This is, in other words, the quantity of public goods which would be optimal if $\tilde{i}$ was missing. If we put $\hat{U}_{\tilde{i}}$ back in it, we get instead g^*. So we can consider the difference $g^* - g_{-\tilde{i}}$ to be *caused* by individual $\tilde{i}$. Then we say "OK dear $\tilde{i}$, with respect to the situation that would have been in place without you, we are spending more in the public good. It is clear that the citizens are benefiting from it, but at the same time that *their* benefit alone does not repay for all the cost of providing the good, since now

$$\sum_{\forall i \neq \tilde{i}} \hat{U}_i'(g) < c.$$



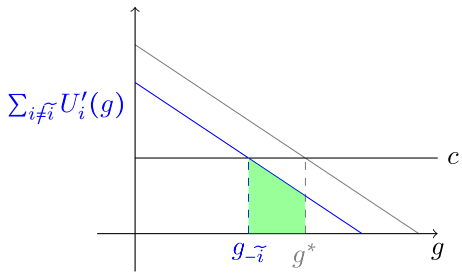
So we will make you pay the difference in welfare." And this difference is composed of two parts: $\tilde{i}$ will have to pay the additional good produced:

$$c \cdot (g^* - g_{-\tilde{i}}) \tag{2}$$



but she will be discounted the additional utility brought to the others:

$$\sum_{i \neq \tilde{i}} \hat{U}_i(g^*) - U_i(g_{-\tilde{i}})$$



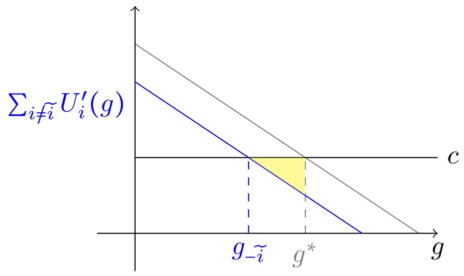
and so finally she will have to pay

$$T_{\tilde{i}} = c \cdot (g^* - g_{-\tilde{i}}) - \sum_{i \neq \tilde{i}} \hat{U}_i(g^*) - \hat{U}_i(g_{-\tilde{i}}):$$

notice this can be rewritten as

$$
\begin{aligned}
T_{\widetilde{i}} &= \int_{g_{-\widetilde{i}}}^{g^*} c \, dg - \sum_{i \neq \widetilde{i}} \int_{g_{-\widetilde{i}}}^{g^*} \hat{U}_i'(g) \, dg \\
&= \int_{g_{-\widetilde{i}}}^{g^*} c - \sum_{i \neq \widetilde{i}} \hat{U}_i'(g) \, dg
\end{aligned}
$$

and is clearly positive (recall (1) and that all U_i's are decreasing).



By repeating those calculations for each individual, we know how much tax to impose on each one of them.

4 Incentive compatibility

Does this really solve the Wicksell problem? Intuitively, let's consider that if $\widetilde{i}$ considers the utilities of the others as given, what she's maximizing by choosing her optimal $\hat{B}_{\widetilde{i}}$ is basically the social welfare.

In fact, let's suppose that $\widetilde{i}$ was thinking of declaring $\hat{B}(x) \equiv 0$, which would have caused the government to buy $g_{-\widetilde{i}}$ of the good. Then, she thinks again, and consider how her utility changes by increasing $\hat{B}(x)$ to the true $B(x)$. There are two effects, fighting each other:

1. the increase in the public good gives her an additional utility, $U_{\widetilde{i}}(g^*) - U_{\widetilde{i}}(g_{-\widetilde{i}})$,
2. but on the other hand the Clarke tax makes her pay for all the welfare loss that the community of the others individuals would incur in if the government provided g^* but $\widetilde{i}$ was absent;

since $\widetilde{i}$ will maximize that additional utility *minus* the tax she will have to pay, we see that the Clarke tax has the feature that it makes $\widetilde{i}$ *maximize the social welfare*. In other terms, it makes $\widetilde{i}$'s interests perfectly aligned with those of the whole society.

Formally, $\widetilde{i}$ solves

$$
\begin{aligned}
\max V(g) &= \max_g U_{\widetilde{i}}(g) - T_{\widetilde{i}} \\
&= \max_g U_{\widetilde{i}}(g) - \left(c \cdot (g - g_{-\widetilde{i}}) - \sum_{i \neq \widetilde{i}} \hat{U}_i(g) - \hat{U}_i(g_{-\widetilde{i}}) \right)
\end{aligned}
$$

and the first order condition for this maximization is

$$\frac{\partial V(g)}{\partial g} = U'_{\widetilde{i}}(g) - c - \sum_{i \neq \widetilde{i}} \hat{U}'_i(g) = 0.$$

Now, if we assume the other individuals reported their utilities truthfully ($\hat{U}_i = U_i$), the above can be rewritten as

$$\sum_i U'_i(g) = c$$

and we know this is the Samuelson condition: we saw above it is satisfied precisely by g^*, and that in order for the government to provide g^* of the good, $\widetilde{i}$ simply has to declare truthfully $\hat{U}_i = U_i$.

So the scheme is incentive-compatible.

5 The Clarke mechanism - complete version

There is still a problem with the above scheme: if we ask to each individual to declare $\hat{U}_i$, they report it and by consequence we provide g^* and make them pay the taxes T_i, then the total amount of taxes collected may be far below the cost of providing g^*.

Collecting a tax for a public good which in the end is not sufficient to repay the public good seems a bit stupid. So we introduce an additional tax: we decide that for any amount g of public good provided, each individual will have to pay $t = \frac{g}{n}$. In other words, given any amount of good, those taxes will be sufficient by themselves to pay the cost of the good. On top of them, the individuals will also have to pay a variable tax, similar to the one in the previous case. Since $\widetilde{i}$ will pay anyway this tax, we redefine $g_{-\widetilde{i}}$ as the amount of public good such that

$$\sum_{\forall i \neq \widetilde{i}} \hat{U}'_i(g_{-\widetilde{i}}) = c - t.$$

and the additional tax as a modified version of (2):

$$T_{\widetilde{i}} = c \cdot (g^* - g_{-\widetilde{i}}) \underbrace{-t(g^* - g_{\widetilde{i}})}_{\text{diff. from (2)}} - \sum_{i \neq \widetilde{i}} \left(\hat{U}_i(g^*) - \hat{U}_i(g_{-\widetilde{i}})\right)$$

(reflecting the fact that $\widetilde{i}$ is *already* paying $t(g^* - g_{\widetilde{i}})$ for the increase in the public good); that's equal to

$$T_{\widetilde{i}} = (n-1)t(g^* - g_{-\widetilde{i}}) - \sum_{i \neq \widetilde{i}} \left(\hat{U}_i(g^*) - \hat{U}_i(g_{-\widetilde{i}})\right).$$

Again, $\widetilde{i}$ is asked to pay the change in welfare for the others, and has to balance that with her own gain in utility from the increase of the good: again, she's optimizing the whole social welfare.

The same maximization above can be used to prove formally that again the optimal strategy is to report her true value truthfully.